On almost rigid rotations

By K. STEWARTSON
Department of Mathematics, University of Bristol

(*Received* 4 *May* 1957)

SUMMARY

In order to answer some of Proudman's questions (1956) concerning shear layers in rotating fluids, a study is made of the flow between two coaxial rotating discs, each having an arbitrary small angular velocity superposed on a finite constant angular velocity. It is found that, if the perturbation velocity is a smooth function of r, the distance from the axis, then the angular velocity of the main body of fluid is determined by balancing the outflow from the boundary layer on one disc with the inflow to the boundary layer on the other at the same value of r. At a discontinuity in the angular velocity of either disc a shear layer parallel to the axis occurs. If the angular velocity of the main body of the fluid is continuous, according to the theory given below the purpose of this shear layer is solely to transfer fluid from the boundary layer on one disc to the boundary layer of the other. It has a thickness $O(\nu^{1/3})$, where ν is the kinematic viscosity, and in it the induced angular velocity is $O(\nu^{1/6})$ of the perturbation angular velocity of the discs. On the other hand, if the angular velocity of the main body of fluid is discontinuous, according to the theory given below the thickness of the shear layer is $O(\nu^{1/4})$. A secondary circulation is also set up in which fluid drifts parallel to the axis in this shear layer and is returned in an inner shear layer of thickness $O(\nu^{1/3})$.

The theory is also applied to the motion of fluid inside a closed circular cylinder of finite length rotating about its axis almost as if solid.

1. INTRODUCTION

Recently Proudman (1956) has investigated the motion of an incompressible viscous fluid confined between two concentric spheres which rotate about a common diameter l with angular velocities Ω and $\Omega(1+\epsilon)$, where ϵ is small. He found that when the kinematic viscosity ν is small the circular cylinder C circumscribing the inner sphere and having its generators parallel to l separates out two regions with different properties. Outside C the fluid rotates as if solid with the angular velocity of the outer sphere while inside C the angular velocity is a function of distance from l only, determined by balancing the fluid entering the boundary layer on the faster moving sphere with the fluid leaving the boundary layer of the slower moving sphere. According to this solution there must be a return of fluid and Proudman inferred that it could occur only near C, but he was unable to decide whether the appropriate shear layer would have a thickness $O(\nu^{1/3})$, $O(\nu^{1/4})$, or even more, and did not determine the behaviour of the fluid in any of these possibilities. In view of the difficulty of determining even the principal properties of rotating fluids in certain problems it is desirable that Proudman's argument be completed so as to eliminate the possibility that the flow between the spheres be of an entirely different character.

The problem considered by Proudman is unfortunately too difficult to solve completely at present, and so in this paper a simpler problem, namely the flow between two coaxial planes rotating almost as if solid, is considered. A complete formal solution may be obtained which supports his inferential argument when ν is small. In addition, if there is a discontinuity at a distance a from the axis in the angular velocity of either discs, a shear layer of the character required by Proudman is formed near the coaxial cylinder of radius a. In this layer any necessary change in the angular velocity of the main body of the fluid occurs in a distance $O(\nu^{1/4})$ while any necessary transfer of fluid from one disc to the other occurs in a distance $O(\nu^{1/2})$. Associated with the change in angular velocity there is also a circulation in which fluid is carried parallel to the axis in a layer of thickness $O(\nu^{1/4})$ and returned in an inner layer of thickness $O(\nu^{1/3})$. A like problem also discussed here is that of two coaxial parallel discs rotating with the same angular velocity but enclosed in a coaxial cylinder of radius a rotating with a slightly different angular velocity. The boundary layer on the cylinder also consists of two parts and on the discs the boundary layer is as usual of thickness $O(\nu^{1/2})$ but only penetrates a distance $O(\nu^{1/4})$ from the cylinder.

2. *Flow between coaxial rotating discs*

Consider two discs in the planes $z = +d$ and $z = -d$ rotating about the z axis with angular velocities

$$\Omega + \Omega\epsilon\{F_1(r) + F_2(r)\} \quad \text{and} \quad \Omega + \Omega\epsilon\{F_1(r) - F_2(r)\},$$

respectively, where ar and az denote distances from an along the axis of rotation, a is a characteristic length and ϵ is small. Let the components of the velocity of the fluid be (u, v, w) relative to fixed cylindrical polar axes with coordinates (r, θ, z). Then since by symmetry all dynamical variables are independent of the azimuthal angle θ we may use the equation of continuity to define a stream function ψ by

$$u = -\frac{\epsilon a\Omega}{r}\frac{\partial\psi}{\partial z}, \qquad w = \frac{\epsilon a\Omega}{r}\frac{\partial\psi}{\partial r}. \tag{2.1}$$

Further write

$$rv = a\Omega r^2 + \epsilon a\Omega\chi. \tag{2.2}$$

If $\epsilon = 0$ the solution of the Navier–Stokes equations which satisfies the boundary conditions is $\chi \equiv \psi \equiv 0$. Hence since ϵ is small it is reasonable to retain only the first order terms in ϵ, when as Proudman has shown the Navier–Stokes equations reduce to

$$2R\frac{\partial\chi}{\partial z} = \left\{\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right\}^2\psi, \tag{2.3}$$

$$-2R\frac{\partial\psi}{\partial z} = \left\{\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right\}\chi, \tag{2.4}$$

where $R = a^2\Omega/\nu$. The boundary conditions are that

$$\left.\begin{aligned} &\psi = \partial\psi/\partial z = 0 \quad \text{at } z = \pm d, \\ &\chi = r^2\{F_1(r) + F_2(r)\} \quad \text{at } z = d, \\ &\chi = r^2\{F_1(r) - F_2(r)\} \quad \text{at } z = -d. \end{aligned}\right\} \tag{2.5}$$

The main interest of the problem stated above lies in its exemplification of the flow arising from more general rotating bodies, such as the spheres considered by Proudman (1956). For rotating discs the particular case when F_1, F_2 are piecewise constant is of greatest interest. It is convenient to treat the problem in two parts by setting F_1 and F_2 successively equal to zero.

3. The antisymmetrical problem

Write

$$\left.\begin{aligned} \chi &= r\int_0^\infty A(k)J_1(kr)\sinh\alpha z\,dk, \\ \psi &= r\int_0^\infty B(k)J_1(kr)\cosh\alpha z\,dk. \end{aligned}\right\} \tag{3.1}$$

These functions are solutions of (2.3) and (2.4) if

$$BR\alpha^2 = -(\alpha^2 - k^2)A, \quad (\alpha^2 - k^2)^3 = -4R^2\alpha^2. \tag{3.2}$$

There are three roots α_1, α_2, α_3, each with positive real parts, of the sextic in α which lead to independent solutions. Hence the general solution of (2.3) and (2.4) of the form (3.1) is

$$\left.\begin{aligned} \chi &= r\sum_{s=1}^{3}\int_0^\infty A_s(k)J_1(kr)\sinh\alpha_s z\,dk, \\ \psi &= r\sum_{s=1}^{3}\int_0^\infty \frac{\alpha_s^2 - k^2}{\alpha_s}A_s(k)J_1(kr)\cosh\alpha_s z\,dk. \end{aligned}\right\} \tag{3.3}$$

Further, from the boundary conditions, since $F_1 = 0$,

$$\left.\begin{aligned} \sum_{s=1}^{3} A_s\sinh\alpha_s d = f_2(k), \quad \sum_{s=1}^{3} A_s(\alpha_s^2 - k^2)\sinh\alpha_s d = 0, \\ \sum_{s=1}^{3} A_s\frac{\alpha_s^2 - k^2}{\alpha_s}\cosh\alpha_s d = 0, \end{aligned}\right\} \tag{3.4}$$

where $$r^2F_2(r) = r\int_0^\infty f_2(k)J_1(rk)\,dk.$$

From the three equations A_s can be determined in terms of f_2, and a complete formal solution found. Of particular interest is the form of the solution when R is large, and during the remainder of the paper we shall concentrate on this special case. Then so long as $k \ll R^{1/2}$ the relevant roots of the sextic (3.2) are

$$\alpha_1 = \frac{k^3}{2R} + O(k^7R^{-3}), \qquad \alpha_2 = R^{1/2}(1+i) + O(k^2R^{-1/2}),$$
$$\alpha_3 = R^{1/2}(1-i) + O(k^2R^{-1/2}). \quad (3.5)$$

It is noted that α_2 and α_3 are associated with regions of rapid change in the axial direction of thickness $O(R^{-1/2})$ while α_1 is associated with a region of rapid change in the radial direction of thickness $O(R^{-1/3})$. It will appear later that all three can combine to give another region of rapid change in the radial direction of thickness $O(R^{-1/4})$. The assumption $k \ll R^{1/2}$ is justified only if the integrands in (3.3) are negligibly small when it is not satisfied. It will appear below that this is true except within a distance $O(R^{-1/2})$ of any irregularity in the angular velocity of either disc.

Further,

$$A_1 = \frac{kf_2}{2R^{1/2}\cosh(k^3d/2R) + k\sinh(k^3d/2R)}, \quad (3.6)$$

$$A_2\exp\{R^{1/2}d(1+i)\} = A_3\exp\{R^{1/2}d(1-i)\} = \frac{2R^{1/2}}{k}A_1\cosh\frac{k^3d}{2R}, \quad (3.7)$$

provided that, in addition to $k \ll R^{1/2}$, $d \gg R^{-1/2}$.

If $F_2(r)$ is a smooth function tending to zero as $r \to \infty$, k^3d may be neglected in comparison with R in the determination of χ and ψ. We find that except when $|z|-d = O(R^{-1/2})$ the contributions from A_2 and A_3 are exponentially small and

$$\chi = \frac{zr}{4R^{3/2}}\int_0^\infty k^4f_2(k)J_1(kr)\,dk \quad (3.8)$$

while

$$\psi = \frac{r}{2R}\int_0^\infty \frac{2R}{k}\left(\frac{kf_2}{2R^{1/2}}\right)J_1(kr)\,dk = \tfrac{1}{2}R^{-1/2}r^2f_2(r). \quad (3.9)$$

In the neighbourhood of $z = \pm d$ there are, in addition to the contribution from A_1, also important contributions from A_2 and A_3 leading to boundary layers of thickness $O(R^{-1/2})$ in which

$$\left.\begin{aligned} \chi &= r^2F_2(r)e^{-x}\cos x, \\ \psi &= \tfrac{1}{2}R^{-1/2}r^2F_2(r)[1-\sqrt{2}\,e^{-x}\cos(x-\tfrac{1}{4}\pi)], \\ x &= (d-|z|)R^{1/2}. \end{aligned}\right\} \quad (3.10)$$

These results agree with Proudman's arrived at inferentially. The outflow from the boundary layer of the slower moving disc must be the same as the inflow to the boundary layer of the faster moving disc. Using Proudman's argument this could only be achieved if the angular velocity in the main body of the fluid were the mean of the angular velocities of the discs at the same value of r, which is in agreement with (3.8).

If $d = O(R^{-1/2})$ a special treatment is required but there is no difficulty. Physically it means that the boundary layers on the discs have joined up.

If F_2 has a discontinuity, at $r = 1$ for example, ψ also has a discontinuity and the integral for χ fails to converge there. A closer examination of the flow near $r = 1$ is necessary and for this purpose let us take

$$\left.\begin{aligned} r^2 F_2 &= -1 \quad (r < 1), \\ &= +1 \quad (r > 1). \end{aligned}\right\} \tag{3.11}$$

Although the same approach as before may be used with success the easiest method is to assume that modifications to the solution already obtained are necessary only in the neighbourhood of $r = 1$. It is then legitimate to neglect the operator $(1/r)\,\partial/\partial r$ in comparison with $\partial^2/\partial r^2$ and $\partial^2/\partial z^2$ in (2.3) and (2.4), and to extend the range of r from $(0, \infty)$ to $(-\infty, \infty)$. The problem is now reduced to a form amenable to the sine transformation. Write

$$\left.\begin{aligned} \chi &= \sum_{s=1}^{3} \int_0^\infty A_s(k) \sin k(r-1) \sinh \alpha_s z \, dk, \\ \psi &= -\frac{1}{2R} \sum_{s=1}^{3} \int_0^\infty \frac{\alpha_s^2 - k^2}{\alpha_s} A_s(k) \sin k(r-1) \cosh \alpha_s z \, dk, \end{aligned}\right\} \tag{3.12}$$

where α_s is defined in (3.2), (3.5), and A_s in (3.4), except that now

$$f_2(k) = 2/\pi k. \tag{3.13}$$

Hence, except when $|z| - d = O(R^{-1/2})$,

$$\chi = \frac{\operatorname{sgn}(r-1)}{\pi} \mathscr{I} \left\{ \int_{-\infty}^{\infty} \frac{e^{ik|r-1|} \sinh(k^3 z/2R)\, dk}{2R^{1/2} \cosh(k^3 d/2R) + k \sinh(k^3 d/2R)} \right\}, \tag{3.14}$$

it being assumed that the integrand is negligibly small when $k = O(R^{1/2})$.

The integral may be evaluated by contour integration on noting that since R is large the poles of the integrand occur at $\cosh(k^3 d/2R) = 0$, and we have

$$\chi = -\frac{2\operatorname{sgn}(r-1)}{3\pi^{2/3} d^{1/3} R^{1/6}} \sum_{n=0}^{\infty} \frac{(-)^n \sin\{(2n+1)\pi z/2d\}}{(2n+1)^{2/3}} \times$$
$$\times [e^{-\beta_n} - 2e^{-\beta_n/2} \cos(\tfrac{1}{2}\sqrt{3}\beta_n - \tfrac{1}{3}\pi)] + O(R^{-1/3}), \tag{3.15}$$

where

$$\beta_n = \left\{\frac{(2n+1)\pi R}{d}\right\}^{1/3} |r-1|.$$

Similarly,

$$\psi = \frac{\operatorname{sgn}(r-1)}{2R^{1/2}} - \frac{2\operatorname{sgn}(r-1)}{3\pi R^{1/2}} \sum_{n=0}^{\infty} \frac{(-)^n \cos\{(2n+1)\pi z/2d\}}{2n+1} \times$$
$$\times [e^{-\beta_n} + 2e^{-\beta_n/2} \cos \tfrac{1}{2}\sqrt{3}\beta_n] + O(R^{-2/3}). \tag{3.16}$$

Near $|z| = d$ extra terms must be added from A_2 and A_3 leading to boundary layers of the type described in (3.10). Thus the thickness of the shear layer near $r = 1$ necessary to permit the return of fluid from one disc to the other is $O(R^{-1/3})$ and in it the fluid velocity is $O(R^{-1/6})$.

The particular form (3.11) was chosen for F_2 because of the simplicity of its sine transform (3.13), but it is to some extent artificial since the angular velocities of the discs are singular at $r = 0$ and in any case the linearization of the equations of motion is invalid there. Nevertheless the solution obtained describes the shear layer appropriate to a discontinuity in the angular velocities of the discs at $r = 1$ because its effect is confined to the immediate vicinity of $r = 1$, and further, outside it,

$$\psi = \tfrac{1}{2}R^{-1/2}\,\mathrm{sgn}(r-1), \qquad \chi = o(R^{-1/6}), \tag{3.17}$$

in agreement with (3.8) and (3.9).

The solution can also be used to complete the description of the flow given in (3.8) and (3.9) in the more realistic case when F_2 is bounded. Let

$$\left.\begin{aligned} F_2 &= l(r) - m(r) \quad (r < 1), \\ &= l(r) + m(r) \quad (r > 1), \end{aligned}\right\} \tag{3.18}$$

where l, m are smooth bounded functions of r, and write χ_0, ψ_0 respectively for the right hand sides of (3.15), (3.16). Then

$$\begin{aligned} \chi &= m(r)\chi_0 + O(R^{-1/3}), \\ \psi &= \tfrac{1}{2}R^{-1/2}r^2 l(r) + r^2 m(r)\,\psi_0 + O(R^{-2/3}) \end{aligned} \tag{3.19}$$

is uniformly valid everywhere except in the boundary layers of the discs.

The effect of the approximations made so far may now be established. The operator $(1/r)\,\partial/\partial r$ makes contributions $O(R^{-1/3}\psi_0)$ to ψ and $O(R^{-1/3}\chi_0)$ to χ. Further, since all the contributions from the shear layer tend to zero exponentially outside it the effect of extending the range of the problem from $0 < r < \infty$ to $-\infty < r < \infty$ is to add exponentially small terms to χ and ψ. Hence in general the solutions given above are the correct leading terms when R is large. There is a breakdown, however, if that part of the range of integration when $k = O(R^{1/2})$ makes a significant contribution to any of the integrals. A simple way of estimating when this is likely to happen is to examine the behaviour of the approximate forms when $k = O(R^{1/2})$. The integrand when $F_2(r)$ is smooth will certainly be negligible in virtue of $f_2(k)$. Let us consider (3.14) as an example of the situation when $F_2(r)$ is not smooth. Writing $k = k_1 + ik_2$, the modulus of the integral when $k_1 > 0$ and $k = O(R^{1/2})$ is

$$\frac{1}{\pi|k+2R^{1/2}|}\exp\left\{-k_1|r-1| - \frac{d-|z|}{R^{1/2}}\,|k_1^3 - 3k_2^2k_1|\right\}$$

and is not exponentially small only when both $|r-1|$ and $d-|z|$ are simultaneously $O(R^{-1/2})$, that is, only in the immediate neighbourhood of the discontinuity in the angular velocity of the discs.

Finally Proudman examined the order of magntidue of the non-linear terms in the Navier–Stokes equations and from his results we deduce that the theory of the flow in the shear layer is valid except at $r = 1$, $|z| = d$ in the limit $\epsilon \to 0$, $R \to \infty$ so that $\epsilon R \to 0$. The condition is more than sufficient to ensure the validity of the theory elsewhere apart from the artificial, but convenient, case described in (3.11) which fails at $r = 0$.

It is possible for $F_1(r)$ to be continuous but to have a discontinuity in its first derivative. An example is

$$r^2F_2(r) = e^{-\gamma|r-1|} \tag{3.20}$$

where γ is a constant. Using the same technique as when F_2 was discontinuous we find that the shear layer is again of thickness $O(R^{-1/3})$ and that in it $\chi = O(R^{-1/2})$, $\psi = O(R^{-5/6})$.

4. The symmetrical problem

In this case

$$\chi = r^2 F_1(r) \quad \text{at } z = \pm d, \tag{4.1}$$

and we write

$$\left.\begin{aligned} \chi &= r \sum_{s=1} \int_0^\infty A_s(k) J_1(kr) \cosh \alpha_s z \, dk \\ \psi &= -\frac{r}{2R} \sum_{s=1}^{3} \int_0^\infty \frac{\alpha_s - k^2}{\alpha_s} A_s(k) J_1(kr) \sinh \alpha_s z \, dk, \end{aligned}\right\} \tag{4.2}$$

and the treatment closely parallels that for the anti-symmetrical problem, the only difference being that cosh and sinh are interchanged throughout. If F_1 is a smooth function it may be shown that

$$\chi = r^2 F_1(r) + O(R^{-1/2}) \quad \text{and } \psi = O(R^{-1}), \tag{4.3}$$

everywhere confirming Proudman's argument. Near the discs there are boundary layers in which the change in χ is $O(R^{-1/2})$ and the change in ψ is $O(R^{-1})$. The flow near a discontinuity however introduces novel features. For defining F_1 as in (3.11),

$$\chi = \frac{\operatorname{sgn}(r-1)}{\pi} \mathscr{I} \left\{ \int_{-\infty}^{\infty} \frac{e^{ik|r-1|} \cosh(k^3 z/2R) \, dk}{2R^{1/2} \sinh(k^3 d/2R) + k \cosh(k^3 d/2R)} \right. \tag{4.4}$$

in the immediate neighbourhood of $r = 1$ except if $|z| - d = O(R^{-1/2})$, when extra terms, as in (3.10), must be added. Since R is large the poles of the integrand occur at $\sinh(k^3 d/2R) = 0$ and at $k^2 d + R^{1/2} = 0$. Evaluating the residues at the relevant poles we have

$$\chi = \operatorname{sgn}(r-1) \Big[1 - \exp\{-R^{1/4} d^{-1/2} |r-1|\} - \\ - \frac{2}{3R^{1/6} d^{1/3}} \sum_{n=1}^{\infty} \frac{(-)^n \cos(n\pi z/d)}{(2n\pi)^{2/3}} \{e^{-\gamma_n} - 2e^{-\gamma_n/2} \cos(\tfrac{1}{2}\sqrt{3}\beta_n - \tfrac{1}{3}\pi)\} \Big] \tag{4.5}$$

where

$$\gamma_n = \left\{\frac{2Rn\pi}{d}\right\}^{1/3} |r-1|. \tag{4.6}$$

Similarly

$$\psi = -\frac{z}{2R^{1/2} d} \operatorname{sgn}(r-1) \exp\{-R^{1/4} d^{-1/2} |r-1|\} + \\ + \frac{\operatorname{sgn}(r-1)}{3\pi R^{1/2}} \sum_{n=1}^{\infty} \frac{(-)^n \sin(n\pi z/d)}{n} [e^{-\gamma_n} + 2e^{-\gamma_n/2} \cos(\tfrac{1}{2}\sqrt{3}\gamma_n)]. \tag{4.7}$$

In this case therefore the shear layer has a thickness $O(d^{1/2} R^{-1/4})$, confirming one of Proudman's suggestions. A shear layer of this type is necessary to allow χ to change sign. As a result fluid drifts from the central plane towards the discs with velocity $O(R^{-1/4})$ and the return of fluid from the discs takes place in an inner boundary layer of thickness $O(R^{-1/3})$ and the azimuthal velocity that is induced is $O(R^{-1/6})$.

As $d \to \infty$ the contribution to χ in (4.3) from the first two terms tends to zero and that from the series is $O(R^{-1/6})$ except in the immediate vicinity of $|z| = d$. Thus if a disc is rotating in an unbounded fluid the angular velocity of the fluid cannot have a finite discontinuity in the limit of zero viscosity, for the shear layer it would generate when the viscosity is small could not be thin. This result is to be contrasted with the related problem of the flow due to the slow motion of a body of revolution along the axis of an unbounded rotating fluid. In the special case of a circular disc of unit radius Morrison & Morgan (1956) have obtained a complete solution. They showed that when ν is small the effect of viscosity may be neglected save near the disc and the cylinder $r = 1$. In the inviscid solution the perturbation of the angular velocity is zero outside and $O\{(1-r)^{-1/2}\}$ inside the cylinder $r = 1$. A shear layer is necessary to smooth out this singularity; it is controlled by the same set of equations as in the problem discussed above and its thickness is $O(R^{-1/3})$. Paradoxically the shear layer can smooth out the infinite discontinuity in Morgan's problem but cannot smooth out the finite discontinuity in the present problem. It would be interesting if Morgan's problem could be solved for a fluid bounded above and below the disc to see whether there is any fundamental change in the character of the shear layer.

A parallel treatment, defining F_2 as in (3.17), shows that, if the angular velocity of the discs has a discontinuous derivative, in the shear layer $\psi = O(R^{-3/4})$ and the increment of χ is $O(R^{-1/4})$.

5. Rotating cylinder

Another example in which one of the friction layers has a thickness $O(R^{-1/4})$ is provided by the flow inside a circular cylinder which is almost rotating like a rigid body. Suppose it has its plane surfaces in the planes $z = \pm d$, the z axis for its axis of symmetry and of rotation, and unit radius. Let the angular velocity of the plane surface be Ω and of the curved surface be $\Omega(1+\epsilon)$. Then proceeding as before we have for boundary conditions

$$\left.\begin{aligned} \psi = \frac{\partial\psi}{\partial z} = \chi = 0 \quad \text{at } z = \pm d \quad (0 < r < 1), \\ \psi = \frac{\partial\psi}{\partial r} = \chi - 1 = 0 \quad \text{at } r = 1 \quad (|z| < d). \end{aligned}\right\} \tag{5.1}$$

It is likely, and will be confirmed later, that the perturbation in the velocity components from those in a state of uniform rotation with angular velocity Ω will be confined to the neighbourhood of $r = 1$. Accordingly we neglect the operator $(1/r)\,\partial/\partial r$ in the linear differential equations satisfied by χ and ψ and extend the range of r to $(-\infty, 1)$.

Then write

$$\chi = \sum_n \sum_{s=1}^{3} A_{sn}\, e^{-k_n(1-r)} \cos \alpha_{sn} z,$$

whence

$$\psi = \frac{1}{2R} \sum_n \sum_{s=1}^{3} A_{sn} \frac{\alpha_{ns}^2 - k_n^2}{\alpha_{sn}}\, e^{-k_n(1-r)} \sin \alpha_{sn} z, \tag{5.2}$$

where α_{sn} is any one of the three roots of

$$(\alpha^2 - k_n^2)^3 + 4R^2\alpha^2 = 0 \tag{5.3}$$

with positive real parts, and the summation is over all acceptable values of k_n. When R is large and $k_n \ll R^{1/2}$ the relevant roots of (5.3) are

$$\alpha_1 = k_n^3/2R, \qquad \alpha_2 = R^{1/2}(1+i), \qquad \alpha_3 = R^{1/2}(1-i). \tag{5.4}$$

The acceptable values of k_n, which must have positive real parts, are determined by the conditions at $|z| = d$. These lead to

$$\left.\begin{aligned} \sum_{s=1}^{3} A_{sn} \cos\alpha_{sn} d = 0, \qquad \sum_{s=1}^{3} \alpha_{sn}^2 A \cos\alpha_{sn} d = 0, \\ \sum_{s=1}^{3} \frac{\alpha_{sn}^2 - k_n^2}{\alpha_{sn}} A_{sn} \sin\alpha_{sn} d = 0, \end{aligned}\right\} \tag{5.5}$$

which have a non-zero solution for A_{sn} only when the determinant of the coefficients vanishes. For large R this means that either

$$\sin(k_n^3 d/2R) = 0, \qquad k_n \neq 0, \quad \text{that is,}$$

$$k_n = \left(\frac{2Rn\pi}{d}\right)^{1/3}, \quad \left(\frac{2Rn\pi}{d}\right)^{1/3}(\tfrac{1}{2}+\tfrac{1}{2}\sqrt{3}i) \quad \text{or} \quad \left(\frac{2Rn\pi}{d}\right)^{1/3}(\tfrac{1}{2}-\tfrac{1}{2}\sqrt{3}i), \tag{5.6}$$

where n is a positive integer, or $k_n = (R/d^2)^{1/4}$. When this condition is satisfied,

$$A_1 \cos\frac{k^3 d}{2R} = -2A_2 \cos R^{1/2}(1+i)d = -2A_s \cos R^{1/2}(1-i)d. \tag{5.7}$$

The contributions from A_2 and A_3 may be neglected except in the immediate neighbourhood of $|z| = d$ and constitute a boundary layer in which the perturbation components in the main body of the fluid are brought to zero at the plane faces. Hence elsewhere, substituting the permissible values of k_n into (5.2), we find that

$$\begin{aligned} \chi = {} & B\exp\{-R^{1/4}d^{-1/2}(1-r)\} + \\ & + \frac{1}{d^{1/3}R^{1/6}} \sum_{n=1}^{\infty} \frac{\cos(n\pi z/d)}{(2\pi n)^{2/3}} \{C_n e^{-\gamma_n} + D_n e^{-\gamma_n/2}\cos(\tfrac{1}{2}\sqrt{3}\gamma_n + \eta_n)\}, \end{aligned} \tag{5.8}$$

$$\begin{aligned} \psi = {} & -\frac{Bz}{2dR^{1/2}} \exp\{-R^{1/4}d^{-1/2}(1-r)\} + \\ & + \frac{1}{2\pi R^{1/2}} \sum_{n=1}^{\infty} \frac{\sin(n\pi z/d)}{n} \{C_n e^{-\gamma_n} + D_n e^{-\gamma_n/2}\cos(\tfrac{1}{2}\sqrt{3}\gamma_n + \eta_n + \tfrac{4}{3}\pi)\}, \end{aligned} \tag{5.9}$$

where γ_n is defined in (4.6), and B, C_n, D_n, η_n are constants. From the boundary conditions at $r = 1$ where $\gamma_n = 0$, we have

$$B = 1, \qquad C_n + C_n \cos\eta_n = 0, \quad (\text{since } \chi = 1), \tag{5.10}$$

$$C_n - D_n \cos\eta_n = O(R^{-1/12}), \quad \left(\text{since } \frac{\partial\psi}{\partial r} = 0\right), \tag{5.11}$$

$$C_n + D_n \cos(\eta_n + \tfrac{4}{3}\pi) = 2(-)^{n+1}, \quad (\text{since } \psi = 0). \tag{5.12}$$

Thus the leading terms are

$$\begin{aligned} \chi = \exp\{-R^{1/4}d^{-1/2}(1-r)\} + \frac{4}{3^{1/2}d^{1/3}R^{1/6}} \sum_{n=1}^{\infty} \frac{(-)^n \cos(n\pi z/d)}{(2\pi n)^{2/3}} \times \\ \times e^{-\gamma_n/2} \sin(\tfrac{1}{2}\sqrt{3}\gamma_n) + O(R^{-1/4}) \end{aligned} \tag{5.13}$$

and

$$\psi = -\frac{z}{2dR^{1/2}}\exp\{-R^{1/4}d^{-1/2}(1-r)\} -$$
$$-\frac{2}{3^{1/2}\pi R^{1/2}}\sum_{n=1}^{\infty}\frac{(-)^n \sin(n\pi z/d)}{n} e^{-\gamma_n/2}\cos(\tfrac{1}{2}\sqrt{3}\gamma_n - 6\pi) + O(R^{-7/12}).$$

Thus again the shear layer near $r = 1$ is in two parts. The main part has thickness $O(R^{-1/4})$ and in it χ is reduced from unity to zero. However to do this ψ and $\partial\psi/\partial r$ must be given non-zero values on the curved surface of the cylinder. An inner boundary layer whose thickness is $O(R^{-1/3})$ is therefore necessary to bring these to zero on the wall. This in turn leads to an inner layer for χ but the change in χ which results is only $O(R^{-1/6})$. Finally near the planes $|z| = d$ additional boundary layers like (3.10) occur to bring the dynamical variables to zero at $|z| = d$. It is noted that these will extend for a distance $O(R^{-1/4})$ inwards from $r = 1$ and $O(R^{-1/2})$ from $|z| = d$ while the basic assumption of this paper, that $k^2 \ll R$ is not justified if $|z| - d$ and $1 - r$ are both $O(R^{-1/2})$.

References

MORRISON, J. A. & MORGAN, G. W. 1956 *Div. Appl. Math. Brown University, Tech. Rep.* no. 8.

PROUDMAN, I. 1956 *J. Fluid Mech.* **1**, 505.